ME 314 - Engineering Design : Mechanical Components
Lecture 7

Note Title

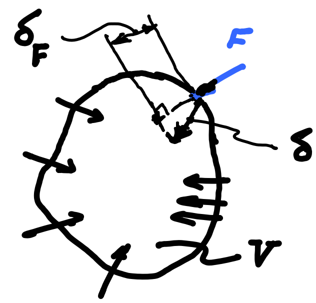
4.11 Castigliano's Method

An energy method called "*Castigliano's Method*" is a powerful and yet simple approach to finding deflection in beams and other structures. This method is based on *Castigliano's Theorem*:

"When a body is elastically deflected by any combination of loads, the deflection at any point and in any direction is equal to the partial derivative of strain energy (computed with all loads acting) w. r. t. a load located at that point and acting in that direction."

For illustration, consider a component under several loads and let the deflection at the point of action of the force F along the line of action of F be δ_F. Then

$$\delta_F = \left.\frac{\partial U}{\partial F}\right|_{F_{final}}$$



where U is the total strain energy stored in the entire body computed with all loads acting, and where the derivative ∂U/∂F is calculated at the final value of F.

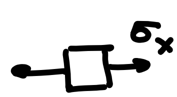
Let u = strain energy density, then

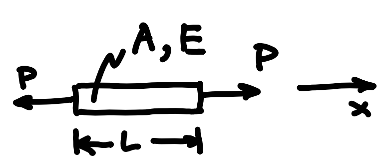
$$[[u]] = \frac{\text{work or energy}}{\text{volume}}$$

$$= \frac{FL}{L^3} = \frac{F}{L^2} \cdot \frac{L}{L}$$

$$= (\text{stress})(\text{strain})$$

For Uniaxial loading





Energy and Deflection Equations for use with Castigliano's Method

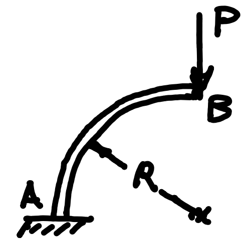
Load Type	Parameters	Starin Energy	If Parameters Do Not Vary With x	General Deflection Equation
Axial	P, E, A	$U=\int_0^L \frac{P^2 dx}{2EA}$	$U=\frac{P^2 L}{2EA}$	$\delta_Q=\int_0^L \frac{P}{EA}\frac{\partial P}{\partial Q}dx$
Bending	M, E, I	$U=\int_0^L \frac{M^2 dx}{2EI}$	$U=\frac{M^2 L}{2EI}$	$\delta_Q=\int_0^L \frac{M}{EI}\frac{\partial M}{\partial Q}dx$
Torsion	T, G, J	$U=\int_0^L \frac{T^2 dx}{2GJ}$	$U=\frac{T^2 L}{2GJ}$	$\delta_Q=\int_0^L \frac{T}{GJ}\frac{\partial T}{\partial Q}dx$
Transverse Shear Rectangular Section	V, G, A	$U=\int_0^L \frac{3V^2 dx}{5GA}$	$U=\frac{3V^2 L}{5GA}$	$\delta_Q=\int_0^L \frac{6V}{5GA}\frac{\partial V}{\partial Q}dx$ *

* Change 6/5 to 1 for quick estimates for non-rectangular sections

If A & E do not vary with x, then

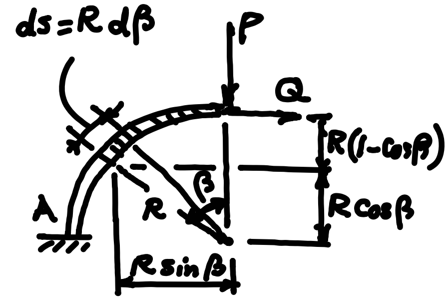
$$U=\frac{P^2 L}{2AE}$$

Example: Determine the horizontal and vertical deflection of point B.



Solution: To find the *Horizontal Deflection at B using the Castigliano's theorem, we need to have a horizontal force acting at point B.*

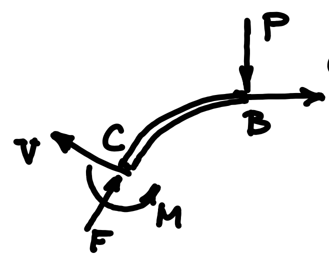
Consider an arbitrary point C. As a result of applying P and Q, an axial force F, a shear force V, and a bending moment M will be developed at C.



Assumption: Energy stored due to transverse shear *V* and tangential (axial) force, *F*, is negligible as compared to *M*.

$$U = \int \frac{M^2 dx}{2EI}, \quad \delta_Q = \frac{1}{EI}\int M \frac{\partial M}{\partial Q}\Big|_{Q=0} dx$$

At point C:



$$M = P.R\sin\beta + QR(1-\cos\beta)$$

$$\frac{\partial M}{\partial Q} = R(1-\cos\beta)$$

$$\delta_Q = \frac{1}{EI}\int_0^{\pi/2} [PR\sin\beta\,(R(1-\cos\beta))]R\,d\beta$$

$$= \frac{PR^3}{EI}\int_0^{\pi/2}(\sin\beta - \sin\beta\cos\beta)\,d\beta$$

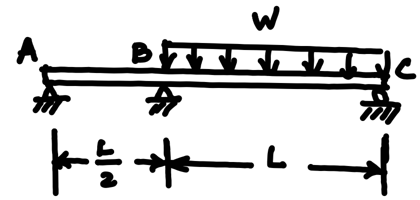
$$= \frac{PR^3}{EI}\left| -\cos\beta + \frac{1}{2}\cos^2\beta \right|_0^{\pi/2} = +\frac{PR^3}{2EI} \Rightarrow$$

Vertical Deflection at B: At C: $M = PR\sin\beta, \quad \frac{\partial M}{\partial P} = R\sin\beta$

$$+\downarrow \delta_P = \frac{1}{EI}\int M \frac{\partial M}{\partial P} ds = \frac{1}{EI}\int_0^{\pi/2} [PR\sin\beta](R\sin\beta)R\,d\beta$$

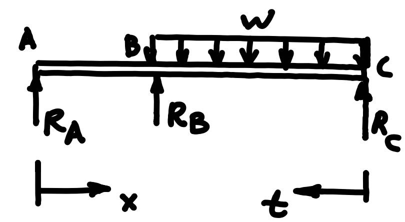
$$= \frac{PR^3}{EI}\int_0^{\pi/2} \sin^2\beta\, d\beta = \frac{PR^3}{EI}\int_0^{\pi/2} \frac{1}{2}(1-\cos 2\beta)\,d\beta$$

$$= \frac{PR^3}{EI}\left| \beta - \frac{1}{2}\sin 2\beta \right|_0^{\pi/2} = \frac{\pi}{4}\frac{PR^3}{EI} \qquad \therefore \delta_P = \frac{\pi}{4}\frac{PR^3}{EI}\downarrow$$



Example 2: For the uniform beam and loading shown, Determine the reactions at each support.

Solution: Consider R_A as " **redundant**", i.e., as if it was an **external load** and compute other reactions in terms of it.



$$\Sigma M_B = 0 \quad : \quad R_C = \frac{wL}{2} + \frac{1}{2} R_A$$

$$\Sigma F_y = 0 \quad : \quad R_B = wL - R_C - R_A = \frac{wL}{2} - \frac{3}{2} R_A$$

Portion AB

$$\left.\begin{array}{l} M = R_A x \\ \dfrac{\partial M}{\partial R_A} = x \end{array}\right\} 0 \leq x \leq \frac{L}{2}$$

Portion BC

$$\left.\begin{array}{l} M = \left(\frac{wL}{2} + \frac{1}{2} R_A\right) t - \frac{1}{2} w t^2 \\ \dfrac{\partial M}{\partial R_A} = \frac{1}{2} t \end{array}\right\} 0 < t < L$$

$$\delta_A = 0 = \frac{1}{EI} \int M \frac{\partial M}{\partial R_A} dx = \frac{1}{EI} \int_0^{\frac{L}{2}} (R_A x)(x) dx + \frac{1}{EI} \int_0^L \left[\left(\frac{wL}{2} + \frac{1}{2} R_A\right) t - \frac{1}{2} w t^2\right] \frac{t}{2} dt$$

or

$$\int_0^{L/2} R_A x^2 dx + \int_0^L \left[\frac{wL}{4} t^2 + \frac{1}{4} R_A t^2 - \frac{1}{4} w t^3\right] dt = 0$$

$$R_A \frac{(L/2)^3}{3} + \frac{wL}{4} \frac{L^3}{3} + \frac{1}{4} R_A \frac{L^3}{3} - \frac{1}{4} w \frac{L^4}{4} = 0$$

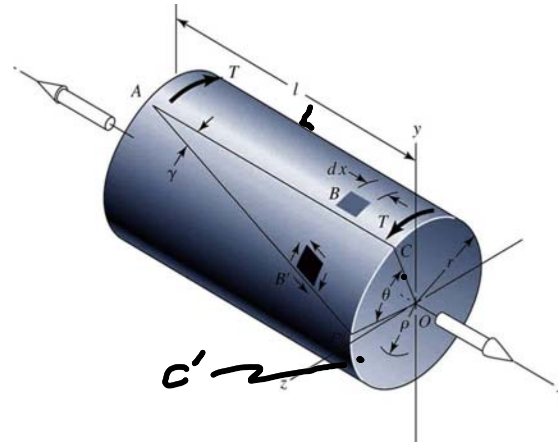
$$R_A L^3 \left(\frac{1}{24} + \frac{1}{12}\right) + wL^4 \left(\frac{1}{12} - \frac{1}{16}\right) = 0 \quad \Rightarrow R_A = -\frac{wL}{6}$$

$$\therefore \underline{R_A = \frac{wL}{6} \downarrow}$$

$$R_C = \frac{wL}{2} - \frac{wL}{12} = + \frac{5wL}{12} \qquad \therefore \underline{R_C = \frac{5wL}{12} \uparrow}$$

$$R_B = \frac{wL}{2} + \frac{3}{2}\left(\frac{wL}{6}\right) = + \frac{3wL}{4} \qquad \therefore \underline{R_B = \frac{3wL}{4} \uparrow}$$

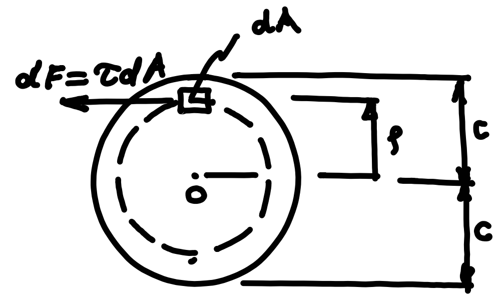
4-12 Torsion



A component loaded with a moment about its longitudinal axis is said to be in torsion, and the applied moment is called a "torque". An important assumption made in the analysis of torsion of shafts with circular cross-section is that cross sections remain plane and perpendicular to the axis (see text, page 177 for other assumptions).

Equilibrium:

$$T = \int_A \rho\, dF = \int_A \rho\, \tau\, dA \qquad (1)$$



Strain:

$$\widehat{CC'} = L\gamma = \rho\theta \qquad (2)$$

$$\gamma = \frac{\rho\theta}{L}$$

Stress-strain Relations:

$$\tau = G\gamma = G\frac{\rho\theta}{L} \qquad (3)$$

Substitute for τ in (1):

$$T = \int_A \rho\left(G\frac{\rho\theta}{L}\right) dA = \frac{G\theta}{L}\int_A \rho^2 dA$$

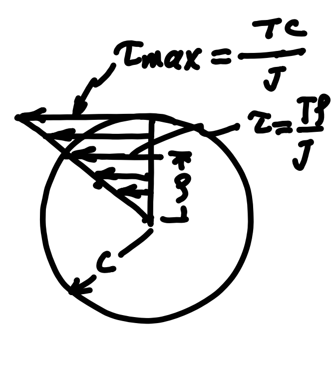
$$\therefore \quad \theta = \frac{TL}{GJ}$$

where

$$J = \int_A \rho^2 dA$$

is the polar moment of area about O. We find

$$J = 2I = \frac{\pi c^4}{2}$$



Substituting for θ in (3):

$$\tau = \frac{T\rho}{J} \quad \therefore \quad \tau_{max} = \frac{Tc}{J} \qquad (4)$$

Hollow shafts with circular cross-section:

$$J = \frac{\pi}{2}\left(c_o^4 - c_i^4\right).$$

Non-circular Cross-sections: Plane sections do not remain plane when the cross-section is not circular. Initially plane sections warp and the shear-stress distribution may not be linear. We have

$$\tau_{max} = \frac{T}{Q} \quad \text{and} \quad \theta = \frac{TL}{KG} \qquad (4\text{-}26)$$

$$\theta = \frac{TL}{(2.25)a^4 G}$$

Q and K are given in Table 4-3 on page 179 of text.

The Black Dots Indicate Points of Maximum Shear Stress (Source: Ref. 4 with Permission)

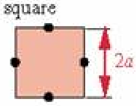
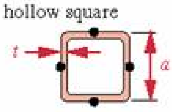
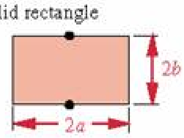
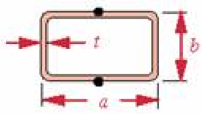
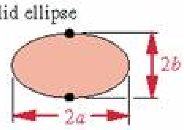
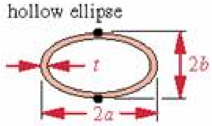
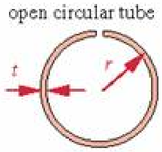
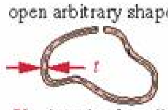
Shape	K	Q
solid square 	$K = 2.25a^4$	$Q = \frac{a^3}{0.6}$
hollow square 	$K = \frac{2t^2(a-t)^4}{2at-2t^2}$ inside corners may have higher stress if corner radius is small	$Q = 2t(a-t)^2$
solid rectangle 	$K = ab^3\left[\frac{16}{3} - 3.36\frac{b}{a}\left(1-\frac{b^4}{12a^4}\right)\right]$	$Q = \frac{8a^2b^2}{3a+1.8b}$
hollow rectangle 	$K = \frac{2t^2(a-t)^2(b-t)^2}{at+bt-2t^2}$ inside corners may have higher stress if corner radius is small	$Q = 2t(a-t)(b-t)$
solid ellipse 	$K = \frac{\pi a^3 b^3}{a^2+b^2}$	$Q = \frac{\pi a b^2}{2}$
hollow ellipse 	$K = \frac{\pi a^3 b^3}{a^2+b^2}\left[1-\left(1-\frac{t}{a}\right)^4\right]$	$Q = \frac{\pi a b^2}{2}\left[1-\left(1-\frac{t}{a}\right)^4\right]$
open circular tube 	$K = \frac{2}{3}\pi r t^3$; $t \ll r$	$Q = \frac{4\pi^2 r^2 t^2}{6\pi r + 1.8t}$; $t \ll r$
open arbitrary shape  U = length of median line	$K = \frac{1}{3}Ut^3$; $t \ll U$ t must be much smaller than minimum radius of curvature	$Q = \frac{U^2t^2}{3U+1.8t}$; $t \ll U$

Table 4-3

Expressions for K and Q for Some Cross-Section Shapes in Torsion.